Noise and synchronization of a single active colloid
Supplementary Note

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This supplementary note presents in more detail the calculation that leads to an estimate of the fluctuations around the synchronized state.

I. CYCLE OF OSCILLATION

We consider the oscillation represented in Fig. 1 of the main text. The initial condition is position $a/2$, following a geometric switch. The time $t_1(i)$ between the geometric switch and the previous clock switch, at cycle i, can be used to measure synchronization. Our aim is to estimate the distribution of $t_1(i+1)$ after one cycle, and use it in a fixed-point argument for the noise.

A cycle corresponds to a sequence of four stages. During the first, the bead has velocity $-v(1-\epsilon)$ during a time $t_2(i) = T_c/2 - t_1(i)$. Subsequently, between the clock switch and the next geometric switch, the velocity is $-v(1+\epsilon)$ during a time $t_3(i)$. The third and fourth stages describe the other half of the oscillation, with (positive) velocity $v(1-\epsilon)$ for a time $T_c/2 - t_3(i)$ and $v(1+\epsilon)$ for a time $t_1(i+1)$.

This calculation requires to evaluate how thermal noise affects the cycle in two ways. First, in the time between a geometric switch and a clock switch, the bead is subject to a driving force and diffusion, which makes the arrival position $r_j(i)$ ($j \in \{1, 2\}$ indexes the half-cycle bead positions) at the clock switch stochastic. Second, the evaluation of the time between the position $r_j(i)$ and the position of the next geometric switch is a first-passage time problem that contributes to the stochasticity of the times $t_3(i)$ and $t_1(i+1)$.

II. EQUATION OF EVOLUTION OF TIME t_1

We now detail the different stages of the cycle. As the delay $t_1(i)$ between the starting geometric switch and the previous clock switch is known, the duration of the random walk between the geometric switch and the next clock switch is prescribed, $t_2(i) = T_c/2 - t_1(i)$. The position of the particle at the clock switch is given by $r_1(i) = v(1-\epsilon)t_2(i) + x_1(i)$. $v(1-\epsilon)t_2(i)$ is the deterministic arrival time and $x_1(i)$ corresponds to the fluctuations due to the diffusion of the particle: $\langle x_1(i)\rangle = 0$ and $\langle x_1^2(i)\rangle = 2Dt_2(i) = 2D\left(T_c/2 - t_1(i)\right)$ with D the diffusion coefficient. In order to carry out the argument, we assume that any transient behavior is past, and that $t_1(i)$ can be approximated by its average t_1^{fp} in the expression of the variance.

After the clock switch, the bead moves over a distance $a' = a - r_1(i)$ at an average velocity $v' = -v(1+\epsilon)$. The arrival time $t_3(i)$ of this process is described by the first-passage time probability density $F(t_3(i))$, where F is the inverse Gaussian [1]

$$F(t) = \frac{1}{\sqrt{4\pi D t^3}} \mathrm{e}^{-\frac{(a'-v't)^2}{4Dt}} \quad . \tag{1}$$

This probability density is well approximated by a Gaussian in the small diffusion limit, i.e. when $a'/v' \ll a'^2/2D$ (in this case the skewness of this distribution is small). Equivalently, this approximation holds for times t such as $v't/a' \in [1-\sqrt{\xi'}, 1+\sqrt{\xi'}]$ with $\xi' = 2D/(a'v')$.

This condition is satisfied for $\xi' \ll 1$, so that the first-passage time distribution F is well-approximated by a Gaussian centered on $(a - r_1(i))/(v(1+\epsilon))$ and of variance $2Da'/(v^3(1+\epsilon)^3)$. Since a' is itself a random variable, we replace, as above in the variance a' by its average value: $\langle a'\rangle = v(1+\epsilon)t_1^{\mathrm{fp}}$. These two assumptions lead to the following formula for the time $t_3(i)$,

$$t_3(i) = \frac{a - r_1(i)}{v(1+\epsilon)} + \zeta_1(i) \tag{2}$$

$$= \frac{A/v - (1-\epsilon)T_c/2}{1+\epsilon} + \kappa t_1 - \frac{x_1(i)}{v(1+\epsilon)} + \zeta_1(i) \tag{3}$$

$$= h(t_1(i)) \ , \tag{4}$$

with $\langle \zeta_1 \rangle = 0$, $\langle \zeta_1^2 \rangle = 2D \langle t_1 \rangle / (v^2(1+\epsilon)^2)$ and $\kappa = (1-\epsilon)/(1+\epsilon)$.

Eq. (4) describes only the first half of the cycle. The second half is symmetric, with the only difference that the velocities become positive. The time $t_1(i+1)$ is therefore obtained by iterating Eq. (4)

$$t_1(i+1) = h(h(t_1(i))) \tag{5}$$

$$= (1+\kappa)\frac{A/v - (1-\epsilon)T_c/2}{1+\epsilon} + \kappa^2 t_1(i) + \chi(i) \ , \tag{6}$$

with

$$\chi(i) = \kappa \left(\frac{x_1(i)}{-v(1+\epsilon)} + \zeta_1(i) \right) + \frac{x_2(i)}{-v(1+\epsilon)} + \zeta_2(i) \ . \tag{7}$$

Here, $x_2(i)$ and $\zeta_2(i)$ represent the fluctuations in the second half of the cycle and are defined similarly as in the first half-cycle. In this approximation, since all the random variables are assumed Gaussian, $\chi(i)$ also follows a Gaussian distribution centered on 0 and with variance

$$\mathrm{var}\,(\chi) = (1+\kappa^2) \left(\frac{\mathrm{var}\,(x_1)}{v^2(1+\epsilon)^2} + \mathrm{var}\,(\zeta_1) \right) \tag{8}$$

$$= (1+\kappa^2)(V_1 + V_2) \ , \tag{9}$$

with $V_1 = 2D(T_c/2 - \langle t_1 \rangle)/(v^2(1+\epsilon)^2)$ and $V_2 = 2D \langle t_1 \rangle / (v^2(1+\epsilon)^2)$.

We assume now that the clock period is set to the natural particle period $2a/v$. The evolution equation over one full cycle becomes

$$\begin{aligned} t_1(i+1) &= (1+\kappa)\frac{A\epsilon}{v(1+\epsilon)} + \kappa^2 t_1(i) + \chi(i) \\ \text{with } \mathrm{var}\,(\chi) &= (1+\kappa^2)\frac{DT_c}{v^2(1+\epsilon)^2} \ . \end{aligned} \tag{10}$$

Removing the noise term $\chi(i)$ in this equation gives the deterministic fixed point $t_1^{\mathrm{fp}} = a/(2v) = T_c/4$. Since the noise only adds fluctuations around this fixed point, we study the deviations $q(i) = t_1(i) - t_1^{\mathrm{fp}}$. Expressed with q, Eq. (10) becomes

$$q(i+1) = \kappa^2 q(i) + \chi(i) \ , \tag{11}$$

or

$$\delta q = q(i+1) - q(i) = -\frac{4q\epsilon}{(1+\epsilon)^2} + \chi(i). \tag{12}$$

Eq. (11) shows the contribution of the deterministic part, due to the system geometry which makes q decay to 0 with the typical decay time of $(1+\epsilon)^2/(4\epsilon)$ cycles, and the noise, which adds fluctuations.

We present now two methods to carry out the argument leading to the quantification of the fluctuations for q.

III. SOLVING USING THE CONTINUOUS LANGEVIN EQUATION

The main text contains an approximated solution in which the main equation is treated as a continuous Langevin equation, although it is discrete. This procedure requires to convert Eq. (12) into a continuous equation. One way to perform this task reducing the size of the steps at each iteration. Formally, we look therefore for an equation

$$q\left(i + \frac{1}{n}\right) = \alpha q(i) + \delta(i) \tag{13}$$

that would describe the system. n is an integer that allows to take the continuum limit. α is a coefficient and $\delta(i)$ a Gaussian random variable. These coefficients can be found by iterating Eq. (13) n times and identifying the coefficients with Eq. (11). The procedure leads to the following evolution equation

$$q\left(i+\frac{1}{n}\right)-q(i)=-(1-\kappa^{\frac{2}{n}})q(i)+\delta(i)\ , \tag{14}$$

with

$$\mathrm{var}\,(\delta)=\frac{1-\kappa^{\frac{4}{n}}}{1-\kappa^4}\mathrm{var}\,(\chi)\ . \tag{15}$$

Eq. (14) is a continuous Langevin equation when n tends to infinity. Therefore, the variance of q at long times is estimated by

$$\mathrm{var}\,(q(\infty))=\lim_{n\to\infty}\frac{\mathrm{var}\,(\delta)}{2(1-\kappa^{\frac{2}{n}})} \tag{16}$$

$$=\lim_{n\to\infty}\frac{1+\kappa^{\frac{2}{n}}}{2(1-\kappa^4)}\mathrm{var}\,(\chi) \tag{17}$$

$$=\frac{\mathrm{var}\,(\chi)}{1-\kappa^4} \tag{18}$$

$$=\frac{DT_c}{4\epsilon v^2}\ . \tag{19}$$

This can be expressed in dimensionless parameters as the fluctuations of the phase ϕ_1,

$$\mathrm{var}\,(\phi_1)=\mathrm{var}\left(\frac{q(\infty)}{T_c}\right)=\frac{\xi}{16\epsilon}\ . \tag{20}$$

When the clock period does not match the natural oscillation time of the oscillator, but is longer or shorter, $T_c=2a/v+\delta$, the calculation follows the same steps, starting form Eq. (10) and replacing the clock period with $T_c=2a/v+\delta$, instead of $2a/v$.

IV. SOLVING BY ITERATING THE EQUATION FOR THE VARIANCE

In order to estimate the fluctuations of q, we can also directly iterate Eq. (11), starting from an initial condition $q(0)$ and

$$q(n+1)=\kappa^{2(n+1)}q(0)+\sum_{i=0}^{n}\kappa^{2i}\chi(n-i)\ . \tag{21}$$

Evaluating the fluctuations of $q(\infty)=\lim_{n\to\infty}q(n+1)$ one gets

$$q(\infty)=\sum_{i=0}^{\infty}\kappa^{2i}\chi(n-i)\ . \tag{22}$$

This is a Gaussian random variable with variance

$$\mathrm{var}\,(q(\infty))=\sum_{i=0}^{\infty}\kappa^{4i}\mathrm{var}\,(\chi) \tag{23}$$

$$=\frac{1}{1-\kappa^4}\mathrm{var}\,(\chi) \tag{24}$$

$$=\frac{DT_c}{4\epsilon v^2}\ . \tag{25}$$

This is the same result as Eq. (19).

[1] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, 2001).